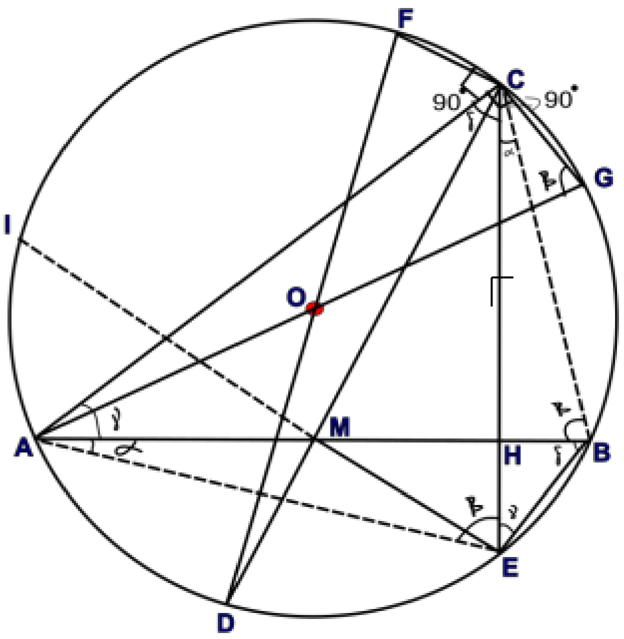
II PRIZE WINNER MRS.A.MADHUMITHA'S SOLUTION FOR 01.01.2023 GEOMETRIC PROBLEM



Given

AB & CE are two chords perpendicular to each other. Let it intersect at H

Let us prove $EB^2 + AC^2 = d^2$

Construct AG as diameter. Join AC, AE, BC, AC.

To show $EB^2 + AC^2 = d^2$ $[\because In\ \Delta AGC, AG^2 = AC^2 + CG^2, d^2 = AC^2 + CG^2]$

It's enough to prove EB = CG

∵AEBC is cyclic quadrilateral

Let $\angle AEC = \angle ABC = \beta, \angle CEB = \angle BAC = \gamma$

$\angle EAB = \angle ECB = \alpha, \angle ABE = \angle ACE = \delta$

In ΔAHE, $\alpha + \beta = 90°$, In ΔAHC, $\gamma + \delta = 90°$

$\alpha + \beta = \gamma + \delta = 90°$

$\because \angle ABC = \angle AGC = \beta$ -------------- (a) ($\because$ $same\ chord\ subtend\ equal\ angles\ at\ circumference$)

$\therefore In\ \Delta AGC,\ \angle ACG = 90°, \angle AGC = \beta$ by (a)

$\angle GAC = \alpha = \angle EAB$

∴ EB=CG

$\Rightarrow EB^2 + AC^2 = d^2$ -------------- (I)

Similarly $AE^2 + BC^2 = d^2$

Also $AH^2 + HB^2 + HE^2 + HC^2 = d^2$ ----------- (II)

(by I as ΔAHE & ΔCHB $are\ right\ angle\ by\ Pythagoras\ theorem$)

In ΔDCF $DF^2 = DC^2 + CF^2 \Rightarrow CF^2 = d^2 - DC^2$ --------------------(1)

$DM^2 = (DC - MC)^2$

$= DC^2 - 2DC \times MC + MC^2$

In ΔDFC, $d^2 = DC^2 + FC^2$

In ΔACG, $d^2 = AC^2 + CG^2$ (or)

$AC^2 + EB^2$ $\{\because CG = EB\}$

$DC^2 + FC^2 = AC^2 + EB^2 = d^2$ --------------------------- (a)

AB & DC intersect at M.

AM $\times MB$ = DM $\times MC$ $\{\because AM = MB \text{ as } M \text{ mid point}\}$

MC x DM = MB^2 --------------------------------- (b)

Consider,

$DM^2 + FC^2 = (DC - MC)^2 + d^2 - DC^2$

$= DC^2 + MC^2 - 2DC \times MC + d^2 - DC^2$

$= MC^2 + d^2 - 2(DM + MC) \times MC$

$= MC^2 + d^2 - 2DM \times MC - 2MC^2$

$= MC^2 + d^2 - 2MB^2 - 2MC^2$ [by (b)]

$= d^2 - 2MB^2 - MC^2$

$= d^2 - 2[MH + HB]^2 - MC^2$

$= d^2 - 2MH^2 - 2HB^2 - 4MH \times HB - MC^2$

$= d^2 - 2MH^2 - 2HB^2 - 4MH \times HB - [HM^2 + HC^2]$

$= d^2 - 3MH^2 - 2HB^2 - 4MH \times HB - HC^2$

$= d^2 - 3MH^2 - 2HB^2 - 4MH \times HB - [BC^2 - HB^2]$

$= d^2 - 3MH^2 - 2HB^2 - 4MH \times HB - BC^2 + HB^2$

$= d^2 - 3MH^2 - 2HB^2 - 4MH \times HB - BC^2$

$= AE^2 - 3MH^2 - HB^2 - 4MH \times HB$

$= AE^2 - [3MH^2 + HB^2 + 4MH \times HB]$ $\{\because d^2 = BC^2 + AE^2\}$

$= AE^2 - (HB + MH)(3MH + HB)$

$= AE^2 - MB(2MH + MB)$

$= AE^2 - MB^2 - 2MB \times MH$

$= AH^2 + HE^2 - MB^2 - 2MB \times MH$

$= (MB + MH)^2 + HE^2 - MB^2 - 2MB \times MH$

$= \cancel{MB^2} + MH^2 + \cancel{2MB \times MH} + HE^2 - \cancel{MB^2} - \cancel{2MB \times MH}$

$= MH^2 + HE^2$

$= ME^2$

Hence $DM^2 + FC^2 = ME^2 \Rightarrow DM, FC$ & ME $are\ sides\ of\ right\ triangle$.